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Relativistic Aharonov–Casher phase in spin 1

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Abstract

The Aharonov–Casher (AC) phase is calculated in relativistic wave equations of spin 1. The AC phase has previously been calculated from the Dirac–Pauli equation using a gauge-like technique. In the spin-1 case, we use Kemmer theory (a Dirac-like particle theory) to calculate the phase in a similar manner. However, the vector formalism, the Proca theory, is more widely known and used. In the presence of an electromagnetic field, the two theories are 'equivalent' and may be transformed into one another. We adapt these transformations to show that the Kemmer theory results apply to the Proca theory. Then we calculate the Aharonov–Casher phase for spin-1 particles directly in the Proca formalism.

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1. Introduction

The Aharonov–Casher (AC) phase is often linked, perhaps subconsciously, with the Aharonov–Bohm (AB) phase. There is a common misconception that the AC phase is the dual of the AB phase, but there are important physical and conceptual differences between them.

For the latter, the absence of any classical field local to the position of the particle makes the force-free nature of the origin of the phase only too evident. For the former, however, the situation is reversed, the particle is all too clearly in the presence of the electric field, and Aharonov and Casher were not too perspicuous about the origins of the force-free effect. Subsequently, much debate has revolved about this issue [3–5].

At an early stage of this debate a concise letter by Goldhaber [3] spelled out the differences between the two effects. These were, in addition to the obvious difference that the particle is located in the field, that:

(a) whereas in the AB effect the flux, although infinitely long, may be curved arbitrarily, in the AC effect the line charge must be straight and parallel to the magnetic moment, and

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(b) there is an extra degree of freedom in the AC effect, namely the spin orientation, which adds a special interest, particularly when considered in the quantum mechanical context.

The first observation, while correct in saying that the force-free phase arises out of a particular configuration of the particle and field, is not yet the full story. It was not until much later [1, 2, 6, 7] that it was noted explicitly that this configuration also required that the magnetic moment of the particle be perpendicular to the plane of its motion.

The necessity of these conditions can be demonstrated by briefly reviewing the relativistic derivation of the phase of He and McKellar in [1]. This derivation has the advantage that the assumptions necessary for the AC phase to appear as an exact result are made explicit, whereas the original derivation, based on the Schrödinger equation used a weak-field approximation which obscured some of the assumptions. The Dirac–Pauli equation of a charge-zero particle with an anomalous magnetic moment in an electromagnetic field[2]

$$\left(\mathrm{i}\gamma^{\mu}\partial_{\mu} + \tfrac{1}{2}\mu\sigma_{\alpha\beta}F^{\alpha\beta} - m\right)\psi = 0 \tag{1}$$

is to be transformed into the free Dirac equation for a wavefunction ψ' which differs from ψ by a phase: $\psi' = \mathrm{e}^{\mathrm{i}\chi}\psi$. The Dirac equation for ψ', when written in terms of ψ,

$$(\mathrm{i}\gamma\cdot\partial - m)\,\mathrm{e}^{\mathrm{i}\chi}\psi = 0 \tag{2}$$

will contain derivatives of the position-dependent phase $\mathrm{e}^{\mathrm{i}\chi}$, which will generate interaction terms in the equation for ψ. For these interactions to have the correct Pauli form the phase χ must be a path-ordered line integral of a field linearly related to the electric field, and must also have an appropriate Dirac matrix dependence. For the phase, make the ansatz

$$\exp\mathrm{i}\chi = \mathcal{P}\exp\mathrm{i}\Gamma\gamma^{0}\mu\int^{\vec{x}}\vec{A}'\,\mathrm{d}\vec{r}\,' \tag{3}$$

where Γ is an appropriate element of the Clifford algebra generated by the Dirac matrices, and $\vec{A}'$ is an effective 'vector potential', related to the electric field, and $\mathcal{P}$ indicates path ordering of the integral in the phase. Both Γ and $\vec{A}'$ are to be determined from the condition that equation (2) generates equation (1).

By eliminating the constant phase component after the operation of the derivative the conditions for the equivalence of (1) and (2) are then

$$\mathrm{e}^{-\mathrm{i}\chi}\gamma^{\nu}\mathrm{e}^{\mathrm{i}\chi}\partial_{\nu}\psi = \gamma^{\nu}\partial_{\nu}\psi \tag{4}$$

$$\mathrm{e}^{-\mathrm{i}\chi}\gamma^{\nu}\mathrm{e}^{\mathrm{i}\chi}\partial_{\nu}\psi = \mu\sigma^{\alpha\beta}F_{\alpha\beta}. \tag{5}$$

Making use of the Baker–Hausdorff formula

$$\mathrm{e}^{-\mathrm{i}\lambda\xi_3}\,\beta^{\mu}\,\mathrm{e}^{\mathrm{i}\lambda\xi_3} = \beta^{\mu} + \mathcal{P}(-\mathrm{i}\lambda)[\xi_3,\beta^{\mu}] + \frac{1}{2!}\mathcal{P}\,(-\mathrm{i}\lambda)^2\,[\xi_3,[\xi_3,\beta^{\mu}]]\ldots \tag{6}$$

the condition (4) can be expressed in terms of a commutation condition,

$$\left[\gamma^{\nu},\Gamma\gamma^{0}\right]\partial_{\nu}\psi = 0 \tag{7}$$

which can be satisfied only if the operator Γ is the product of two spatial gamma matrices, say $\Gamma = \gamma^1\gamma^2$, *and* the wavefunction is independent of the spatial coordinate corresponding to the gamma matrix which does not appear in this equation. In this case, $\partial_3\psi = 0$. For future use we note that we may rewrite[3]

$$\mathrm{i}\Gamma\gamma^{0} = \gamma_5\gamma^{3} \tag{8}$$

[2] We consider the case where the field $F^{\mu\nu}$ is a pure electric field, with $F^{0i} = E_i$ being the only non-vanishing components.

[3] We define $\gamma_5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3$. Note we will also omit further reference to path ordering for simplicity.

which is the operator which has as eigenstates those states with well defined components of the spin in the 3-direction, when the motion of the particle is normal to the 3-direction [8].

The choice of the operator Γ, with equation (2), then means that the term $\gamma^i\gamma^1\gamma^2\gamma^0 A'_i$ must be $\sigma^{0j}E_j$, i.e.

$$A'_1 = -E_2 \qquad A'_2 = E_1 \tag{9}$$

which may be written conveniently as

$$A'_i = -\varepsilon_{ij}E_j \tag{10}$$

where ε_{ij} is the two-component antisymmetric tensor with the value $\varepsilon_{12} = +1$.

This derivation clearly shows that the dynamics are restricted to 2 + 1 dimensions: the conditions $\partial_\perp\psi = 0$, $E_\perp = 0$ and $\partial_\perp E_\perp = 0$, apply where ψ is an eigenstate of $\sigma_\perp$, and the subscript $\perp$ indicates the coordinate perpendicular to the plane of motion [1].

The wavefunction ψ is related to the solution of the free equation (2) by a well defined, coherent phase. Therefore, we infer that it does not alter any kinematic properties of the solution. The interaction thus isolated can therefore be referred to as 'force-free'. A particle with zero charge but with an anomalous magnetic moment will accumulate a topological phase, when a strict configuration of velocity, magnetic moment and the line of charge is satisfied. This will be referred to as the AC configuration.

Because these conditions have been shown to lead to the AC phase without approximation only for a Dirac particle, yet we know from the original non-relativistic approach that they lead to the AC phase in the weak-field approximation for arbitrary spin, we ask whether it is possible to extend the exact derivation to relativistic higher-spin systems. Here we take the first step in providing an answer to that question by showing that the derivation of [1] can indeed be extended to the relativistic spin-1 particle.

2. Aharonov–Casher phase in spin 1

2.1. The Kemmer theory

This method exploits the genuine first-order nature of the Dirac theory, where the derivative of the path-dependent phase produced terms linear in the field strengths, and hence comparable to the elements of the Dirac–Pauli interaction term. So naturally we extend this method to spin 1 using the first-order Kemmer spinor theory [9][4].

The 16-dimensional spin-1 spinor ϕ has a Dirac-like free equation of motion,

$$\left(\mathrm{i}\beta^\mu\partial_\mu - m\right)\phi = 0 \tag{11}$$

where the β-matrices are generalizations of the Dirac gamma matrices. These satisfy an algebra ring, which for spin 1 is

$$\beta^\lambda\beta^\mu\beta^\nu + \beta^\nu\beta^\mu\beta^\lambda = \eta^{\lambda\mu}\beta^\nu + \eta^{\mu\nu}\beta^\lambda. \tag{12}$$

These Kemmer β-matrices are reducible, that is the 16×16 representation decomposes into three separate representations: a one-dimensional trivial representation; a 5D spin-0 representation; and the 10D spin-1 representation. Of course these each satisfy (12) separately. It is also noteworthy that this algebra ring is 'odd', that is it cannot reduce the matrix operator to identity, unlike the Dirac algebra.

[4] This spin-1 object in $SL(2, c)$ was first investigated by Duffin, Kemmer and Petiau [9]. Periodic interest in the formalism continued. Further descriptions of Kemmer theory can be found in [11], Fischbach *et al* and Vijayalakshmi *et al* [9, 10]. An application to cross section calculations can be seen in Knurth Kerr *et al* [9].

Just as in Dirac theory, the Lorentz invariance of the Kemmer theory entails a transformation of the spinor so that the matrix representation remains the same. The Lorentz generator for these transformations, $S_{\mu\nu}$, like its spin-$\frac{1}{2}$ equivalent $\sigma_{\mu\nu}$, is proportional to the antisymmetric product of two matrices of the ring:

$$S_{\mu\nu} = b\left(\beta_\mu\beta_\nu - \beta_\nu\beta_\mu\right).$$

These generators satisfy well known commutation relations (see [10, 11]) and define the spin operators. The coefficient b is linked to the coefficient in the Lorentz transformation, and hence to the coefficient of the commutation relations, and is set below according to our convenience.

The equation of motion of a spin-1 neutral particle with an anomalous magnetic moment in Kemmer theory is

$$\left(\mathrm{i}\beta^\mu\partial_\mu + \tfrac{1}{2}\mu S_{\alpha\beta}F^{\alpha\beta} - m\right)\phi = 0. \tag{13}$$

The interaction term does emerge from the derivation of a 'second-order Kemmer equation' following the method of Umezawa [11, 12], however, we rely on the reduction of this term to its non-relativistic equivalent to confirm this form as well as to fix the ambiguity in coefficients outlined above. In particular, this is done so that the sought-for phase can be compared with its $\frac{1}{2}$ equivalent. We also note in advance that this term is transformed into its equivalent in Proca theory [10].

The operator component of the phase in the spin-$\frac{1}{2}$ AC phase solution was found to be a pseudo-vector spin operator. This operator can be rewritten as the antisymmetric product of three matrices of the Dirac algebra, which suggests that we adopt a spin-1 pseudo-vector operator defined by

$$\xi_\mu = \tfrac{1}{2}\mathrm{i}\varepsilon_{\mu\nu\lambda\rho}\beta^\nu\beta^\lambda\beta^\rho. \tag{14}$$

This is easily verified to be a spin operator in the rest frame.

Now a path-dependent phase proportional to ξ_3 is introduced in the free Kemmer equation of motion (11),

$$\left(\mathrm{i}\beta^\mu\partial_\mu - m\right)\exp\left[\mathrm{i}\xi_3\int^r \vec{A}'\,\mathrm{d}r\right]\phi = 0 \tag{15}$$

with the intention of transforming this into the equation of motion (13) with the anomalous magnetic moment term. Allowing the operators to act on the solution ϕ', and eliminating the phase supplies two conditions:

$$\exp\left[-\mathrm{i}\xi_3\int^r \vec{A}'\,\mathrm{d}r\right]\beta^\mu\exp\left[\mathrm{i}\xi_3\int^r \vec{A}'\,\mathrm{d}r\right] = \beta^\mu \tag{16}$$

and

$$-\beta^\mu\xi_3 A'_\mu\phi = \tfrac{1}{2}\mu S_{\alpha\beta}F^{\alpha\beta}\phi = \mu S_{0l}F^{0l}\phi. \tag{17}$$

Making use of the Baker–Hausdorff formula, the first condition reduces to the requirement that the commutator be zero. If $\mu \neq 3$ then this is automatically satisfied, however, for $\mu = 3$, inspection of the definition of ξ_3 and the β-algebra will show that the commutator does not vanish. Thus in order to satisfy the first condition the dynamics of the system are restricted to $2 + 1$ dimensions, just as for spin-$\frac{1}{2}$ in [1]. In particular, $\partial_3\phi$ and A'_3 are zero.

Of the second condition, consider first the operators of the left-hand side. For $\mu = 0$ we obtain a solution which corresponds to F_{12}, which for the sake of this calculation is zero. For $\mu = 1, 2$ we have (repeated indices are no longer summed):

$$-\beta^\mu\xi_3 = \beta_\mu\xi_3 = -\epsilon_{\mu\nu}\tfrac{1}{2}S_{0\nu}\beta_\mu^2 \quad \left(= -\epsilon_{\mu\nu}\tfrac{1}{2}\beta_\mu^2 S_{0\nu}\right) \tag{18}$$

where $\epsilon_{\mu\nu}$ is an antisymmetric tensor with $\epsilon_{12} = 1$. Then for $\mu = 1, 2$ the action of β_μ on (18) gives

$$-\tfrac{1}{2}\eta_{\mu\mu}\epsilon_{\mu\nu}\beta_\mu S_{0\nu} = +\tfrac{1}{2}\epsilon_{\mu\nu}\beta_\mu S_{0\nu}. \tag{19}$$

Employing this, the definition of ξ_3 and the β-algebra,

$$s_3 S_{0\nu}\beta_\mu^2\phi_s = -S_{0\nu}s_3\phi_s \tag{20}$$

where the states ϕ_s are eigenstates of the spin operator ξ_3. Then it follows that $S_{0\nu}\beta_\mu^2\phi_s$ equals $-S_{0\nu}\phi_s$, hence for the operators of the left-hand side of (17) acting on ϕ_s equal $\frac{1}{2}\epsilon_{\mu\nu}S_{0\nu}\phi_s$ for any value of s_3. Now the comparison of these terms with the those of the Dirac–Pauli interaction term on the right-hand side of (17) yields the field conditions:

$$A'_1 = -2\mu E_2 \qquad A'_2 = 2\mu E_1 \qquad E_3 = 0. \tag{21}$$

Clearly, given that $E_3 = 0$, and adding the further restriction that $\partial_3 E_3 = 0$, then the curl of $\vec{A}'$ is equal to $2\mu\nabla\cdot\vec{E}$. When the particle is moved around a closed path in this configuration the AC phase is given by

$$\phi_{AC} = \xi_3 \oint \vec{A}'\,\mathrm{d}r = 2\mu\xi_3 \int_S (\nabla\cdot\vec{E})\,\mathrm{d}S = 2\mu\xi_3\lambda \tag{22}$$

where λ is the line density of charge. The conditions for the AC effect with spin-1 particles are exactly those for spin $\frac{1}{2}$, except that the spin operator and spinor have changed. The factor of two shows that the phase is twice that accumulated by a spin-$\frac{1}{2}$ particle with the same magnetic moment coupling constant, in the same field.

2.2. *Equivalence of AC phase in Proca theory*

For the description of spin-1 particles, however, the Kemmer theory is not as widely known nor used as the Proca wave equations. Moreover, these two theories are equivalent in the sense that, including the electromagnetic interactions, they can be transformed into one another [10]. Therefore, we should like to demonstrate that the results obtained above are the same as those obtainable in Proca theory. However, in Proca theory which, being inherently a second-order theory, produces derivatives of the fields which are not present in the electromagnetic interaction at this order, this phase technique 'fails'. So as a first step we demonstrate this equivalence by the transformation of the AC phase obtained in the Kemmer theory into the Proca formalism.

The transformation from the Kemmer to the Proca formalism is usually introduced in the context of selecting projection operators which pick out the spin-1 irreducible representation from the β-matrices [10]. These operators then reproduce the Proca equation. The projection matrices U^μ and $U^{\mu\nu}$ are constructed from the β-matrices:

$$U^\mu = -(\beta^1)^2(\beta^2)^2(\beta^3)^2(\beta^\mu\beta^0 - \eta^{\mu 0}) \tag{23}$$

$$U^{\mu\nu} = U^\mu\beta^\nu = -U^{\nu\mu} \tag{24}$$

$$U^\mu\beta^\nu\beta^\sigma = \delta^{\nu\sigma}U^\mu - \delta^{\mu\sigma}U^\nu. \tag{25}$$

The four vector components of the Proca theory ψ^ν, and six field components $G^{\mu\nu}$ are defined, respectively, by the relations

$$U^\nu\phi = \mathrm{i}\sqrt{m}\psi^\nu \qquad U^{\mu\nu}\phi = \frac{1}{\sqrt{m}}G^{\mu\nu}.$$

Then the operation of $U^{\mu\nu}$ on the Kemmer equation yields the antisymmetric field tensor expressed in terms of the vector components,

$$G^{\mu\nu} = \partial^{\mu}\psi^{\nu} - \partial^{\nu}\psi^{\mu} \tag{26}$$

while the operation of U^{ν} on the Kemmer equation yields the Proca equation:

$$\partial_{\mu}G^{\mu\nu} + m^2\psi^{\nu} = 0. \tag{27}$$

At this stage we merely comment whether the Kemmer spinor is modified by a simple (c-number, path-independent) phase, $e^{i\lambda}\phi$, then these transformations produce a Proca vector, $e^{i\lambda}\psi^{\nu}$, and field tensor, $e^{i\lambda}G^{\mu\nu}$, similarly modified by this phase.

Earlier we indicated that the anomalous magnetic moment term which in equation (13) was expressed in terms of the Lorentz operator $S_{\mu\nu}$ contains an ambiguity. Therefore, in the following, it is convenient to drop the notation of equation (13) and express the anomalous magnetic moment in its 'absolute' form using the commutator of β-matrices

$$i\mu\left[\beta^{\mu}, \beta^{\nu}\right] F_{\mu\nu}.$$

The transformation of this term yields $-2im\mu\left(F_{\sigma}{}^{\mu}\psi^{\sigma}\right)$. This can be compared with that derived from the introduction of minimal coupling in the Proca equations or with the standard anomalous interaction form:

$$i\mu m\hat{I}^{\mu\nu}{}_{\alpha\beta}F_{\mu\nu}\psi^{\beta} = i\mu m\left(g^{\mu}_{\ \alpha}g^{\nu}_{\ \beta} + g^{\mu}_{\ \beta}g^{\nu}_{\ \alpha}\right)F_{\mu\nu}\psi^{\beta} = -2i\mu m F_{\beta\alpha}\psi^{\beta}.$$

The factor of m is introduced so that the magnetic moment μ has the usual dimensions $[\mu] = -1$.

As noted above, while the spin operator ξ_{μ} commutes with the β matrices for unequal indices, this is not true when the indices are equal. Consequently, ξ_{μ} cannot commute with the transformation matrices U^{μ} (and therefore $U^{\mu\nu} = U^{\mu}\beta^{\nu}$), because U^{μ} contains all of the β matrices at least once. Therefore, we cannot simply derive the equivalent free Proca equation with a spin-dependent phase.

However, in the case that the fields are projected onto specific spin states, the action of the pseudo-spin operator ξ_3 on the Kemmer field ϕ yields the same eigenvalue s_3 as the spin operator $\tilde{S}_3$ acting on the Proca fields ψ^{μ} and $G^{\mu\nu}$ derived from that Kemmer spinor using the relations given below. Therefore, the magnitude/sign of the phase acquired in the Kemmer formalism is the same as that in the Proca formalism. Then the c-number phase $e^{i\lambda s_3}\phi$ which modifies the Kemmer spinor does, as noted above, apply to both the Proca vector and the field tensor, and we can then generalize to the operator phase $e^{i\lambda\tilde{S}_3}$ without a penalty for inexactness, under the standard conditions for the Aharonov–Casher effect.

Speaking in general terms, the Kemmer equation can be thought of as an operator on the Kemmer wavefunction

$$O_K[\boldsymbol{A}]\phi_K = 0$$

and the transformation operator $\mathcal{T}$ is defined such that the ten independent Proca fields $\psi_P = \mathcal{T}\phi_K$. Then for a spin-dependent phase

$$\mathcal{T}e^{i\lambda\Sigma_3}\phi_{Kfree} = \mathcal{T}e^{i\lambda s_3}\phi_{Kfree} = e^{i\lambda s_3}\psi_{Pfree} = e^{i\lambda\hat{S}_3}\psi_{Pfree}. \tag{28}$$

In order to show this we choose a specific representation for the 10-component Kemmer spinor $\phi = (\phi_1, \phi_2, \ldots, \phi_{10})^T$, and the related β-matrices. Our choice of the spinor components in terms of the Proca 4-vector ψ^{μ} and the Proca field strength $G^{\mu\nu}$ is exactly that given by Greiner [10], equation (15.43). Then the Proca equations can be written in the Kemmer form as

$$\left(\mathrm{i}\beta_\mu \partial^\mu - m\right)\phi = 0$$

where the 10×10 matrices are

$$\beta^0 = \begin{pmatrix} \varnothing & \varnothing & \mathbb{1} & \bar{O}^\dagger \\ \varnothing & \varnothing & \varnothing & \bar{O}^\dagger \\ \mathbb{1} & \varnothing & \varnothing & \bar{O}^\dagger \\ \bar{O} & \bar{O} & \bar{O} & 0 \end{pmatrix} \qquad \beta^k = \begin{pmatrix} \varnothing & \varnothing & \varnothing & -\mathrm{i}K^{k\dagger} \\ \varnothing & \varnothing & S^k & \bar{O}^\dagger \\ \varnothing & -S^k & \varnothing & \bar{O}^\dagger \\ -\mathrm{i}K^k & \bar{O} & \bar{O} & 0 \end{pmatrix} \tag{29}$$

where the elements are

$$\varnothing = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \mathbb{1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$S^1 = \mathrm{i}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \qquad S^2 = \mathrm{i}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \qquad S^3 = \mathrm{i}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{30}$$

$$K^1 = (1 \quad 0 \quad 0) \qquad K^2 = (0 \quad 1 \quad 0) \qquad K^3 = (0 \quad 0 \quad 1)$$

$$\bar{O} = (0 \quad 0 \quad 0).$$

It should be remarked that these β matrices differ from those given by Greiner [10].

The spin operator for the Proca vector is given by

$$S_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} & 0 \\ 0 & \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{31}$$

Then from this, for $s_3 \neq 0$, $(s_3)^2 = 1$, we deduce $\psi^1 = -\mathrm{i}s_3\psi^2$, $\psi^0 = \psi^3 = 0$. The tensor components dependent on these are G^{01}, $G^{02} = \mathrm{i}s_3 G^{01}$, G^{23}, $G^{13} = -\mathrm{i}s_3 G^{23}$, and

$$G^{12} = -\mathrm{i}s_3 \partial_\mu \psi^\mu = 0.$$

The condition on the polarization 4-vector that $s^\mu p_\mu = 0$, requires, for a spin orientation in the 3-direction, $\partial^3\psi^\mu = 0$. This with the condition that $\psi^3 = 0$, gives $G^{23} = 0$. As S_3 commutes with momentum operators we can express all these relations in the eigenfunction:

$$\tilde{S}_3 \begin{pmatrix} G^{01} \\ \mathrm{i}s_3 G^{01} \\ 0 \\ G^{23} \\ \mathrm{i}s_3 G^{23} \\ 0 \\ \psi^1 \\ \mathrm{i}s_3\psi^1 \\ 0 \\ 0 \end{pmatrix} = s_3 \begin{pmatrix} G^{01} \\ \mathrm{i}s_3 G^{01} \\ 0 \\ G^{23} \\ \mathrm{i}s_3 G^{23} \\ 0 \\ \psi^1 \\ \mathrm{i}s_3\psi^1 \\ 0 \\ 0 \end{pmatrix}$$

with the 10×10 spin matrix $\tilde{S}_3$,

$$\tilde{S}_3 = \begin{pmatrix} S_3 & \varnothing & \varnothing & \bar{O}^\dagger \\ \varnothing & S_3 & \varnothing & \bar{O}^\dagger \\ \varnothing & \varnothing & S_3 & \bar{O}^\dagger \\ \bar{O} & \bar{O} & \bar{O} & 0 \end{pmatrix}. \tag{32}$$

In the Kemmer theory the spin operators are $\Sigma_i = \mathrm{i}\varepsilon_{ijk}\left[\beta_j, \beta_k\right]$. The Σ_3 matrix is easily confirmed to be S_3, i.e. the spin matrices from the two theories are identical. What, then, of the pseudo-spin operator ξ_3? Explicitly $\xi_3 = \mathrm{i}\left(\beta^0\beta^1\beta^2 + \beta^1\beta^2\beta^0 + \beta^2\beta^0\beta^1\right)$. So this matrix is

$$\xi_3 = \begin{pmatrix} \varnothing & \varnothing & S_3 & \bar{O}^\dagger \\ \varnothing & \varnothing & \varnothing & -\mathrm{i}K^{3\dagger} \\ S_3 & \varnothing & \varnothing & \bar{O}^\dagger \\ \bar{O} & \mathrm{i}K^3 & \bar{O} & 0 \end{pmatrix} \tag{33}$$

which does not look like Σ_3. However, the operator is evaluated as (see [10, 11])

$$\beta^0\xi_3 = \xi_3\beta^0 = \begin{pmatrix} S_3 & \varnothing & \varnothing & \bar{O}^\dagger \\ \varnothing & \varnothing & \varnothing & \bar{O}^\dagger \\ \varnothing & \varnothing & S_3 & \bar{O}^\dagger \\ \bar{O} & \bar{O} & \bar{O} & 0 \end{pmatrix} \tag{34}$$

which is almost Σ_3. The absent element operates (in the Proca case) on the components proportional to G^{23}, which vanish as we saw. The action of Σ_3 or $\tilde{S}_3$ on the appropriate column is then equivalent to that of $\xi_3\beta^0\phi$.

This then demonstrates that the individual spin eigenstates of the two representations are transformed into one another. That means that the spin operator in the phase may be temporally replaced with its eigenvalue before performing the transformation between representations and then replaced with its counterpart. In this way the second element of the task of manifesting the transference of the AC phase from the Kemmer theory to the Proca formalism can be realized. A spin-1 object acquires the same phase in either theory when an equivalent magnetic moment interaction term is used. The problem of deriving the AC phase in the Proca formalism directly is the subject of the next section.

2.3. Calculation of the AC phase in Proca theory

The Aharonov–Casher phase can be derived for the Proca vector using this phase technique, albeit with the conditions discovered earlier. Again we propose to identify the solution of the Proca equation with the anomalous magnetic moment interaction with a phase-modified free Proca field. It was seen in section 2.2 that the transformation of the Kemmer spinor showed that both the Proca vector and the Proca field were modified by the same phase (in spite of the fact that one is the derivative of the other): $\psi'^\nu = \exp\left(\mathrm{i}\hat{S}_3 \int \vec{A}'\,\mathrm{d}\vec{r}\right)\psi^\nu$, and $G'^{\mu\nu} = \exp\left(\mathrm{i}\hat{S}_3 \int \vec{A}'\,\mathrm{d}\vec{r}\right)G^{\mu\nu}$. Then the vector ψ^ν satisfies both

$$\partial_\mu G^{\mu\nu} - 2\mathrm{i}\mu m F_\mu{}^\nu\psi^\mu + m^2\psi^\nu = 0 \tag{35}$$

with its subsidiary conditions, and

$$\partial_\mu G'^{\mu\nu} + m^2\psi'^\nu = 0 \tag{36}$$

also with its subsidiary conditions.

At this stage it is convenient to define the vector $\mathcal{F}^\nu$ as the vector term of the magnetic moment interaction

$$\mathcal{F}^\nu = F_\mu{}^\nu \psi^\mu = \begin{pmatrix} -E_1\psi^1 - E_2\psi^2 - E_3\psi^3 \\ -E_1\psi^0 - B_3\psi^2 + B_2\psi^3 \\ -E_2\psi^0 + B_3\psi^1 - B_1\psi^3 \\ -E_3\psi^0 - B_2\psi^1 + B_1\psi^2 \end{pmatrix}.$$

Then for spin projection ± 1 eigenstates with $B_k = E_3 = 0$ the interaction term $-2\mathrm{i}\mu m\mathcal{F}^\nu$ becomes

$$-2\mathrm{i}\mu m\mathcal{F}^0 = 2\mathrm{i}\mu m\left(E_1\psi^1 + E_2\psi^2\right). \tag{37}$$

This is the expression to be derived from equation (36). An immediate difficulty in this task is that this equation is inherently a second-order equation. The action of the derivative on the path-dependent phase leaves a remnant term containing the field tensor, which is the first derivative of the Proca vector. This derivative can be eliminated using a relation now derived from the subsidiary equations.

By contracting ∂_ν with equations (35) and (36), respectively, gives

$$m\partial_\nu\psi^\nu = 2\mathrm{i}\mu\partial_\nu\mathcal{F}^\nu \tag{38}$$

$$0 = \exp\left[\mathrm{i}s_3 \int \vec{A}'\,\mathrm{d}\vec{r}\right]\left\{\mathrm{i}s_3\left(\partial_\nu A'_\mu\right)G^{\mu\nu} + m^2\partial_\nu\psi^\nu + \mathrm{i}s_3 m^2 A'_\nu\psi^\nu\right\}. \tag{39}$$

So if the vector ψ^μ is to be a simultaneous solution of both equation (35) and (36), then it also satisfies the subsidiary conditions

$$\partial_\nu\psi^\nu = 2\mathrm{i}\frac{\mu}{m}\partial_\nu\mathcal{F}^\nu = -\mathrm{i}\frac{s_3}{m^2}\left\{\left(\partial_\nu A'_\mu\right)G^{\mu\nu} + m^2 A'_\nu\psi^\nu\right\}. \tag{40}$$

Now if the fields A'_μ satisfy the conditions $A'_0 = \partial_0 A'_\mu = 0$, then the first term of the second condition vanishes, and we have the useful relation

$$2\mathrm{i}\mu\partial_\nu\mathcal{F}^\nu = -\mathrm{i}s_3 m A'_\nu\psi^\nu. \tag{41}$$

Now the expansion of equation (36) as seen above leaves us attempting to make the remaining term $\mathrm{i}s_3 A'_\mu G^{\mu\nu}$ look like the anomalous magnetic moment interaction (37). At this point it is assumed that the phase fields A'_k are the same as those of section 2.1. It is also assumed that ψ^μ is a spin eigenstate with $(s_3)^2 = 1$, so that $\psi^1 = -\mathrm{i}s_3\psi^2$, etc. Then, by substitution of the electric and Proca field identities:

$$\mathrm{i}s_3 A'_\mu G^{\mu\nu} = 2\mu m\partial^0\mathcal{F}^0. \tag{42}$$

We can take advantage of the fact that $\mathcal{F}^i = 0$, to replace $\partial^0\mathcal{F}^0$ with $\partial_\mu\mathcal{F}^\mu$, and then introduce the identity (41). This gives $2\mu m\partial^0\mathcal{F}^0 = 2\mu m\partial_\mu\mathcal{F}^\mu = -s_3 m A'_\mu\psi^\mu$. Now the last term gives us (repeating the substitution of the electric and Proca field identities),

$$s_3 m\left(A'^1\psi^1 + A'^2\psi^2\right) = 2\mathrm{i}\mu\left[E_1\psi^1 + E_2\psi^2\right] \tag{43}$$

as the zeroth element of our vector. This is the interaction term (37) which was sought. Naturally the integral of the path-dependent phase $\hat{S}_3 \oint \vec{A}'\,\mathrm{d}\vec{r}$ equals the same phase as the Kemmer calculation, as that is how we have chose the field relations.

Hence the AC phase can be calculated successfully by the gauge method in the tensor formalism, although with the assistance of the relations first derived from the spinor theory, and is seen to be consistent between the two spin-1 theories.

3. Remarks

The topological nature of the Aharonov–Casher effect is delicate and its appreciation requires a subtle understanding. These derivations of the topological phase from spin-$\frac{1}{2}$ and spin-1 relativistic wave equations illuminate some of the difficulties at the intersection of quantum mechanics and classical electromagnetic theory, and those associated specifically with the AC effect.

Foremost are the explicit constraints on the geometrical alignment between the electric field, the particle momentum and the spin quantization axis. That these constraints are obscured in the non-relativistic derivation of the AC effect is evident by their protracted revelation in the literature. In contrast, they evolve out of the relativistic derivation in an obvious manner because of the need to reduce the number of spinor components which can contribute to a geometric, scalar phase. The implications of such strict limitations on the interpretation of experiments which have sought to measure the Aharonov–Casher phase are self-evident.

Furthermore, these studies indicate the extraordinary care which must be taken in applying semiclassical models to spin or the magnetic moment, or even the particle's trajectory. The natural determination of a spinor's properties out of the transformation properties of the $SL(2, C)$ group contrasts with the difficult interpolation of spinor behaviour in a non-relativistic framework. In the case of the AC effect, satisfaction of the geometrical constraints for a geometric phase is dependent on the choice of interpretation of the classical vector which substitutes for the spin dynamics. In many cases a phase can still be derived from a classical origin which is mistakenly thought to be the geometric phase. Similarly, the problem of the 'hidden momentum' of a magnetic dipole is clarified by comparison with the non-relativistic equations derived from the complete relativistic wave equations.

Finally, the existence of the AC phase for all the spin components of a spin 1 particle, together with the geometrical constraints, mark out the differences between the Aharonov–Bohm and Aharonov–Casher effects, and underscore the limitations of 'equivalence' methods of studying the AC effect using the former. Equivalence methods can only map two possible spin states onto the charge states of the Aharonov–Bohm effect, and consequently omit part of the richness of the AC effect.

4. Conclusion

The Aharonov–Casher phase has been successfully calculated for spin-1 relativistic particles in both relativistic theories with equivalent electromagnetic interactions. In the Kemmer theory, the definition of a pseudo-vector spin operator ξ_μ has permitted the imitation of the gauge technique used by He and McKellar in spin $\frac{1}{2}$ to find the AC phase, subject to the conditions familiar from spin $\frac{1}{2}$. In order to demonstrate that these results hold in the more familiar Proca wave theory, the two equations, one with the interaction, the other with the modified phase, have been shown to transform into their equivalents using the established projection operators. Finally, with the conditions derived from the previous studies, the Aharonov–Casher phase has been calculated directly in the Proca formalism itself. The coherence of these results demonstrates the validity of the spin dependence in the AC effect for higher spins.

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